So, the area swept out is equal to K times the time it takes to sweep it out!

Which law have we just verified?

Kepler's Second Law!

We call K the Kepler constant of the orbit.

Now we turn our focus to converting

$$mr\frac{d^2x}{dt^2} = -F_x \;\&\; mr\frac{d^2y}{dt^2} = -F_y \quad (**)$$

to polar.

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\cos\theta - r\sin\theta\cdot\frac{d\theta}{dt}\right)$$

$$= \frac{d^2r}{dt^2}\cos\theta - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dr}{dt}\sin\theta\frac{d\theta}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}$$

$$= \frac{d^2r}{dt^2}\cos\theta - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - \sin\theta\left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)$$

Because $(r(t))^2\theta'(t) = 2k$,

$$0 = \frac{d}{dt}(2k) = \frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 2r\frac{dr}{dt}\frac{d\theta}{dt} + r^2\frac{d^2\theta}{dt^2}$$

So, we have $\dfrac{d^2x}{dt^2} = \cos\theta\left[\dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2\right]$

Similarly,

$$\frac{d^2y}{dt^2} = \sin\theta\left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]$$

Putting these into $(**)$ gives:

$$F\cos\theta = m\cos\theta\left[r\left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2}\right] \quad \text{and}$$

$$F\sin\theta = m\sin\theta\left[r\left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2}\right]$$

(after cancelling the r's on both sides)

Since $\sin\theta$ & $\cos\theta$ are not simultaneously zero:

$$F = m\left[r\left(\frac{d\theta}{dt}\right)^2 - \frac{d^2r}{dt^2}\right]$$

$$r^2 \frac{d\theta}{dt} = 2k \Rightarrow \frac{d\theta}{dt} = \frac{2k}{r^2}$$

So, finally

$$\boxed{F(t) = m\left[\frac{4k^2}{r(t)^3} - \frac{d^2 r}{dt^2}\right]}$$

We have succeeded in expressing $F(t)$ in terms of the mass m of P, Kepler's constant k, and the distance $r = r(t)$ from O to P.

Conic Shaped Orbit (Trajectory) $\Rightarrow$ Inverse Square Force

Let's assume that the trajectory of the particle is a conic section

$$r = f(\theta) = \frac{d}{1+\varepsilon\cos\theta}$$

(the center of the centripetal force is a focus of the conic)

Let's write

$$r = d(1+\varepsilon\cos\theta)^{-1}$$

(r & θ are functions of t)

Then

$$\frac{dr}{dt} = -d(1+\varepsilon\cos\theta)^{-2}(-\varepsilon\sin\theta)\frac{d\theta}{dt}$$

$$= \varepsilon d(1+\varepsilon\cos\theta)^{-2}\sin\theta\left(\frac{2k}{r^2}\right)$$

Now, since $\dfrac{1}{r^2} = \dfrac{(1+\varepsilon\cos\theta)^2}{d^2}$

the equation becomes

$$\frac{dr}{dt} = 2k\sin\theta\,\cancel{(1+\varepsilon\cos\theta)^{-2}}\cdot\varepsilon\cancel{d}\cdot\frac{\cancel{(1+\varepsilon\cos\theta)^2}}{d^{\cancel{2}}}$$

$$= \frac{2k\varepsilon}{d}\sin\theta$$

Taking another derivative, we get

$$\frac{d^2r}{dt^2} = \frac{2k\varepsilon}{d}\cos\theta\,\frac{d\theta}{dt} = \frac{2k\varepsilon}{d}\cos\theta\left(\frac{2k}{r^2}\right)$$

$$= \frac{4k^2\varepsilon}{d}\cos\theta\cdot\frac{1}{r^2}$$

Now, plug this into the formula we found for F

$$F = m\left(\frac{4k^2}{r^3} - \frac{d^2r}{dt^2}\right) = m\left(\frac{4k^2}{r^3} - \frac{4k^2\varepsilon}{d}\cos\theta\cdot\frac{1}{r^2}\right)$$

$$= 4mk^2\left(\frac{1}{r} - \frac{\varepsilon}{d}\cos\theta\right)\cdot\frac{1}{r^2}$$

But,

$$\frac{1}{r} = \frac{1+\varepsilon\cos\theta}{d}$$

Thus, we have

$$F = 4mk^2\left(\frac{1}{d} + \cancel{\frac{\varepsilon\cos\theta}{d}} - \cancel{\frac{\varepsilon\cos\theta}{d}}\right)\cdot\frac{1}{r^2}$$

$$= \frac{4mk^2}{d}\cdot\frac{1}{r^2}$$

Which is an inverse square force.

Inverse Square Force $\Longrightarrow$ Conic Trajectory

Newton's Law of Universal Gravitation

$$F = G\frac{Mm}{r^2}$$

G = gravitational constant
M = mass of object
m = mass of other object
r = distance between

Let's write $C = GM$ and choose the object with mass M to be at the origin.

Comparing this force with our previous formula, we get